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LETTER TO THE EDITOR

Operator content of the Ashkin-Teller quantum chain—superconformal and Zamolodchikov-Fateev invariance: I. Free boundary conditions

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Abstract. Based on our numerical analysis, we conjecture the operator content of the finite-size limit of the spectra of the Ashkin-Teller model with free boundary conditions. The same operator content is obtained from a Hamiltonian with a four-fermion interaction and a U(1) Kac-Moody Sugawara structure.

For some special values of the coupling constant the model exhibits $N = 2$ superconformal and Zamolodchikov-Fateev invariance. The operator content in these cases is expressed in terms of irreducible representations of the corresponding algebras.

The quantum version of the Ashkin-Teller (1943) model was introduced by Kohmoto *et al* (1981), who considered the Hamiltonian:

$$H = \frac{1-4h}{4\sqrt{\lambda}\, h \sin \pi/4h} \sum_{j=1}^{N} [(\sigma_j + \sigma_j^3 + \varepsilon\sigma_j^2) + \lambda(\Gamma_j\Gamma_{j+1}^3 + \Gamma_j^3\Gamma_{j+1} + \varepsilon\Gamma_j^2\Gamma_j^2)] \tag{1}$$

where N represents the number of sites, λ plays the role of the inverse of temperature, ε is a coupling constant $(-1 \leqslant \varepsilon \leqslant 1)$,

$$h = \frac{\pi}{4\cos^{-1}(-\varepsilon)} \tag{2}$$

and the matrices σ and Γ are

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3}$$

As will become clear in a short while it is more convenient to consider h instead of ε as the parameter of the model. Thus $h = \frac{1}{4}$ corresponds to the four-state Potts model, $h = \frac{1}{2}$ to two decoupled Ising models, $h = 1$ to a Kosterlitz-Thouless transition and $h = \infty$ to a frozen phase. The model is self-dual and has a continuous phase transition at $\lambda = 1$ for $\infty > h \geqslant \frac{1}{4}$. For $h > 1$ the model exhibits a critical fan: for each h it stays critical in a domain $1/\lambda_{\max}(h) \leqslant \lambda \leqslant \lambda_{\max}(h)$. In the whole region of criticality the phase transition corresponds to a Virasoro algebra with a central charge $c = 1$ (von Gehlen and Rittenberg 1987). Since our numerical studies suggest that in the critical fan the critical exponents depend only on h and not on λ we will consider here only the half-line $\lambda = 1$, $h \geqslant \frac{1}{4}$.

The aim of this letter† is to describe the finite-size limit of the spectrum of the Hamiltonian (1) with free boundary conditions:

$$\Gamma_{N+1} = 0. \tag{4}$$

Here we describe only the main results of our work, as an extended version is going to be published elsewhere. The spectra for boundary conditions compatible with the torus are shown in the following letter (Baake *et al* 1987).

The finite-size sscaling limit of the spectrum of the Hamiltonian is defined as follows. Let $E_k(N)$ be the energy levels ($E_0(N)$ corresponds to the ground-state energy) for the Hamiltonian with N sites. We consider the quantities (Cardy 1984, 1986, von Gehlen and Rittenberg 1986a):

$$\mathscr{E}_k = \lim_{N\to\infty} \frac{N}{\pi}(E_k(N) - E_0(N)). \tag{5}$$

It is a consequence of conformal invariance in two dimensions that the quantities $\mathscr{E}_k$ are described by unitary irreducible representations (IR) of the Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{m+n} + \tfrac{1}{12}cn(n^2-1)\delta_{n,-m} \tag{6}$$

where n, $m \in \mathbb{Z}$ and $c = 1$. An IR characterised by the highest weight Δ

$$L_0|\Delta\rangle = \Delta|\Delta\rangle \qquad L_n|\Delta\rangle = 0 \qquad (n \geqslant 1) \tag{7}$$

gives a contribution

$$\mathscr{E}_r = \Delta + r \qquad (r = 0, 1, 2, \ldots) \tag{8}$$

to the spectrum $\mathscr{E}_k$. Δ is a *surface critical exponent* and the level $\Delta + r$ has a degeneracy $D(\Delta, r)$. For $c = 1$, which is our case, D is independent of Δ and equal to the function $\pi(r)$ determined by the partition function:

$$\pi_V(q) = \prod_{m=1}^{\infty} \frac{1}{1-q^m} = \sum_{r=0}^{\infty} \pi(r)q^r \tag{9}$$

unless $\Delta = \frac{1}{4}t^2$ where t is an integer. In this case $D(\frac{1}{4}t^2, r)$ is determined by the partition function (Kac 1979)

$$(1-q^{t+1})\pi_V(q) = \sum_{r=0}^{\infty} D(\tfrac{1}{4}t^2, r)q^r. \tag{10}$$

The level Δ is called *ascendant*, the levels $\Delta + r$ *descendants*. Since the Hamiltonian (1) is parity invariant it is important to see the relative parity of the descendants and the ascendant. From the outer automorphism

$$L_n \mapsto (-1)^n \qquad n \in \mathbb{Z} \tag{11}$$

of the Virasoro algebra (6) we learn that the levels $|\Delta + r\rangle$ have a relative parity $(-1)^r$ to the level $|\Delta\rangle$. From now on we denote by $(\Delta)^P$ ($P = \pm$) the conformal tower (ascendant plus descendants) corresponding to an ascendant with parity P.

As the reader might have noticed, the relation (8) is valid only for a special normalisation of the Hamiltonian (1) (which corresponds to taking the 'sound velocity' equal to one). From our numerical analysis we have guessed the proper normalisation factor given in (1).

† Some results from this letter have been presented earlier (Rittenberg 1986).

Before we can display the spectra we have to know the symmetry of the problem. With the exception of the point $h=\frac{1}{4}$, where the symmetry is higher, the model has D_4 symmetry. This is the group of order eight corresponding to the transformations:

$$(\Gamma_j')^m = A^{mn}(\Gamma_j)^n \tag{12}$$

in the Hamiltonian (1). The eight matrices A^{mn} are

$$\Sigma^l = \begin{pmatrix} \mathrm{i}^l & 0 & 0 \\ 0 & \mathrm{i}^{2l} & 0 \\ 0 & 0 & \mathrm{i}^{3l} \end{pmatrix} \qquad (l=0,1,2,3) \qquad C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{13}$$

and $\Sigma^l C$. We recall that D_4 has a two-dimensional representation:

$$D(\Sigma^l) = \begin{pmatrix} \mathrm{i}^l & 0 \\ 0 & \mathrm{i}^{-l} \end{pmatrix} \qquad D(C) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{14}$$

and four one-dimensional representations $D_{\alpha,\beta}(a, \beta = 0, 1)$:

$$D_{\alpha,\beta}(\Sigma) = (-1)^\beta \qquad D_{\alpha,\beta}(C) = (-1)^\alpha. \tag{15}$$

An extensive study of the lower part of the spectrum which has extended our previous results (von Gehlen and Rittenberg 1986b, 1987) has let us to conjecture the following operator content for the various sectors (irreducible representations of D_4) of the spectrum:

$$\begin{aligned} &D_{0,0}\colon \bigoplus_{k\geqslant 0} (4k^2)^+ \oplus \bigoplus_{k\geqslant 1} \left(\frac{4k^2}{h}\right)^+ \\ &D_{1,0}\colon \bigoplus_{k\geqslant 0} \left(\frac{(2k+1)^2}{h}\right)^+ \\ &D_{0,1}\colon \bigoplus_{k\geqslant 0} ((2k+1)^2)^+ \oplus \bigoplus_{k\geqslant 1} \left(\frac{4k^2}{h}\right)^- \\ &D_{1,1}\colon \bigoplus_{k\geqslant 0} \left(\frac{(2k+1)^2}{h}\right)^+ \\ &D\colon \bigoplus_{k\geqslant 0} \left(\frac{(2k+1)^2}{4h}\right)^+. \end{aligned} \tag{16}$$

(The parities in (16) are defined always relative to the lowest level within each sector which is taken, by convention, to have parity +.) One should keep in mind that, since the D representation is two dimensional, the corresponding operator content appears twice in the spectrum. Notice that some anomalous dimensions are dependent on the coupling constant h and some are not. We also observe that the anomalous dimensions are all positive for the whole interval $h>0$. From now on we will assume that there exists such a model where also the region $\frac{1}{4}>h>0$ is displayed. Equations (16) have been checked at the Ising decoupling point ($h=\frac{1}{2}$) and one can see that they have the proper S_4 symmetry at the four-state Potts model point ($h=\frac{1}{4}$).

In order to study the properties of the system under modular transformations (Cardy 1986), it is interesting to compute the partition functions:

$$Z_g = \mathrm{Tr}(g\, q^{L_0}) \tag{17}$$

where g is one of the eight group elements of D_4. Obviously Z_g is not a function of the group elements but of the conjugacy classes. For the five conjugacy classes of D_4

$$\begin{array}{lll} \mathrm{I}: \{e = \Sigma^0\} & \mathrm{II}: \{\Sigma^2\} & \mathrm{III}: \{\Sigma, \Sigma^3\} \\ \mathrm{IV}: \{\Sigma C, \Sigma^3 C\} & \mathrm{V}: \{C, \Sigma^2 C\} & \end{array} \tag{18}$$

we have the following partition functions:

$$\begin{aligned} Z_{\mathrm{I}} &= \vartheta_3(0, q^{1/4h})\pi_{\mathrm{V}}(q) \\ Z_{\mathrm{II}} &= \vartheta_4(0, q^{1/4h})\pi_{\mathrm{V}}(q) \\ Z_{\mathrm{III}} &= Z_{\mathrm{IV}} = \vartheta_4(0, q)\pi_{\mathrm{V}}(q) \\ Z_{\mathrm{V}} &= \tfrac{1}{2}(Z_{\mathrm{II}} + Z_{\mathrm{III}}) \end{aligned} \tag{19}$$

where

$$\vartheta_3(0, q) = \sum_{n\in\mathbb{Z}} q^{n^2} \qquad \vartheta_4(0, q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}. \tag{20}$$

The operator content of the finite-size scaling limit spectrum (16) can be obtained from the following fermionic Sugawara construction of a U(1) Kac-Moody algebra (Corrigan 1986 and references therein). Consider two sets of fermionic operators a_μ and b_ν ($\mu, \nu \in \mathbb{Z}+\frac{1}{2}$):

$$\begin{array}{lll} \{a_\mu, \alpha_\nu\} = \delta_{\mu,-\nu} & \{b_\mu, b_\nu\} = \delta_{\mu,-\nu} & \\ [a_\mu, b_\nu] = 0 & a_\mu^+ = a_{-\mu} & b_\mu^+ = b_{-\mu} \end{array} \tag{21}$$

and the Kac-Moody generators:

$$\begin{array}{ll} T_m = \sum_\mu a_{m-\mu} b_\mu & m \in \mathbb{Z}, m \neq 0, \mu \in \mathbb{Z}+\frac{1}{2} \\ T_0 = \dfrac{1}{\sqrt{2h}} \sum_\mu a_{-\mu} b_\mu & L_m = \frac{1}{2}\sum_n :T_{m-n}T_n: \\ (L_m^+ = L_{-m}, T_n^+ = T_{-n}, m, n \in \mathbb{Z}) & \end{array} \tag{22}$$

where :: denotes the normal product. The L_m generators verify the Virasoro algebra (6) with $c = 1$ and together with the T_n they define the U(1) Kac-Moody algebra:

$$[T_m, L_n] = mT_{m+n} \qquad [T_m, T_n] = m\delta_{m,-n} \qquad m, n \in \mathbb{Z}. \tag{23}$$

The L_0 generator given by (22) has the spectrum (16). The IR of a U(1) Kac-Moody algebra are given by an ascendant Δ and descendants $\Delta + r$ with a degeneracy $\pi(r)$ (see (9)) for *any* Δ (including $\Delta = 0$). We now notice that we can combine the IR of the Virasoro algebra appearing in $D_{0,0}$ and $D_{0,1}$

$$\{0\} = \bigoplus_{k\geq 0} (4k^2) \qquad \{1\} = \bigoplus_{k\geq 0} ((2k+1)^2) \tag{24}$$

into one IR corresponding to $\Delta = 0$ of the U(1) Kac-Moody algebra:

$$(0)^{\mathrm{KM}} = \{0\} \oplus \{1\} \tag{25}$$

and thus the entire spectrum (16) can be described in terms of IR of the U(1) Kac-Moody algebra.

We now consider the problem of higher symmetries (infinite Lie algebras or superalgebras) in the Ashkin–Teller model. An inspection of (16) shows that, in general, the spectrum is described by an infinite number of ascendants. The situation is different for special values of h. There the spectrum can be described by a *finite* number of ascendants (primary fields) but with a degeneracy of the descendants larger than $\pi(r)$. This occurs when some of the ascendants of Virasoro IR collapse on the descendants of some other IR. This phenomenon probably occurs for any h being a rational number. Here we explore only some special cases. We start with $N=1$ superconformal invariance. The corresponding superalgebra (Friedan *et al* 1985, Berdshadski *et al* 1985, Eichenherr 1985) is given by (6) together with

$$\begin{aligned}&[L_m, G_r]=(\tfrac{1}{2}m-r)G_{m+r}\\&\{G_r, G_s\}=2L_{r+s}+\tfrac{1}{3}c(r^2-\tfrac{1}{4})\delta_{r_1-s}\end{aligned}\tag{26}$$

where $r,\ s\in\mathbb{Z}$ in the Ramond sector and $r,\ s,\ \in\mathbb{Z}+\frac{1}{2}$ in the Neveu–Schwarz sector. The possible values of Δ for $c=1$ are:

$$\begin{aligned}&\text{Ramond:}\qquad (\tfrac{1}{24})_1^{\mathrm{R}}(\tfrac{1}{16})_1^{\mathrm{R}}(\tfrac{3}{8})_1^{\mathrm{R}}(\tfrac{9}{16})_1^{\mathrm{R}}\\&\text{NS:}\qquad (0)_1^{\mathrm{NS}}(\tfrac{1}{16})_1^{\mathrm{NS}}(\tfrac{1}{6})_1^{\mathrm{NS}}(1)_1^{\mathrm{NS}}.\end{aligned}\tag{27}$$

From the Goddard *et al* (1986) character expression one can derive the following decomposition of the IR of the $N=1$ superconformal algebra in terms of IR of the Virasoro algebra:

$$\begin{aligned}
&(0)_1^{\mathrm{NS}}=[0]_1\oplus[\tfrac{3}{2}]_1 &&(1)_1^{\mathrm{NS}}=[1]_1\oplus[\tfrac{3}{2}]_1\\
&(\tfrac{1}{16})_1^{\mathrm{NS}}=[\tfrac{1}{16}]_1\oplus[\tfrac{9}{16}]_1 &&(\tfrac{1}{6})_1^{\mathrm{NS}}=[\tfrac{1}{6}]_1\oplus[\tfrac{2}{3}]_1\\
&[0]_1=\{0\}\oplus\bigoplus_{k\geqslant 1}(6k^2) &&[1]_1=\{1\}\oplus\bigoplus_{k\geqslant 1}(6k^2)\\
&[\tfrac{3}{2}]_1=\bigoplus_{k\geqslant 0}(\tfrac{3}{2}(2k+1)^2) &&[\tfrac{1}{16}]_1=(\tfrac{1}{16})_1^{\mathrm{R}}=\bigoplus_{k\in\mathbb{Z}}(\tfrac{1}{16}(8k+1)^2)\\
&[\tfrac{9}{16}]_1=(\tfrac{9}{16})_1^{\mathrm{R}}=\bigoplus_{k\in\mathbb{Z}}(\tfrac{1}{16}(8k+3)^2) &&[\tfrac{1}{6}]_1=\bigoplus_{k\in\mathbb{Z}}(\tfrac{1}{6}(6k+1)^2)\\
&[\tfrac{2}{3}]_1=\bigoplus_{k\in\mathbb{Z}}(\tfrac{2}{3}(3k+1)^2)\\
&(\tfrac{1}{24})_1^{\mathrm{R}}=\bigoplus k\in\mathbb{Z}\ (\tfrac{1}{24}(6k+1)^2) &&(\tfrac{3}{8})_1^{\mathrm{R}}=\bigoplus_{k\geqslant 0}(\tfrac{3}{8}(2k+1)^2).
\end{aligned}\tag{28}$$

A close examination of (16) shows that there are four values of h where the spectra can be expressed in terms of IR of $N=1$ (strong hints in this direction are already known, see von Gehlen and Rittenberg (1986b) and Friedan and Shenker (1986)). The operator content in each of those cases is

$$\begin{aligned}
&h=\tfrac{1}{6}: && D_{0,0}\oplus D_{1,1}=[0]_1 \qquad D_{0,1}\oplus D_{1,0}=[1]_1 \qquad D=[\tfrac{3}{2}]_1\\
&h=\tfrac{3}{2}: && D_{0,0}\oplus D_{1,1}=[0]_1\oplus[\tfrac{2}{3}]_1 \qquad D_{0,1}\oplus D_{1,0}=[1]_1\oplus[\tfrac{2}{3}]_1 \qquad D=[\tfrac{3}{2}]_1\oplus[\tfrac{1}{6}]_1\\
&h=\tfrac{2}{3}: && D_{0,0}=[0]_1 \qquad D_{0,1}=[1]_1 \qquad D_{1,0}=D_{1,1}=[\tfrac{3}{2}] \qquad D=(\tfrac{3}{8})_1^{\mathrm{R}}\\
&h=6: && D_{0,0}=[0]_1\oplus[\tfrac{2}{3}]_1 \qquad D_{0,1}=[1]_1+[\tfrac{2}{3}]_1 \qquad D_{1,0}=D_{1,1}=[\tfrac{3}{2}]_1\oplus[\tfrac{1}{6}]_1\\
& && D=(\tfrac{3}{8})_1^{\mathrm{R}}\oplus(\tfrac{1}{24})_1^{\mathrm{R}}.
\end{aligned}\tag{29}$$

Notice that the Neveu-Schwarz multiplets appear with the proper Z_2 grading. The existence of the $(0)_1^{NS}$ and $(1)_1^{NS}$ IR for all the four values of h and the fact that the $(\frac{1}{16})^{NS}$, $(\frac{1}{16})^{R}$, $(\frac{9}{16})^{R}$ IR do not appear is a signal that the symmetry of the problem is even higher. The symmetry is given by the $N=2$ superconformal algebra (Di Vecchia *et al* 1985, Waterson 1986, Boucher *et al* 1986) which is obtained combining together the U(1) Kac-Moody algebra (already known to be present in the system for any h) with the $N=1$ superconformal algebra which is there for the four above-mentioned values of h.

Another interesting algebra which shows up in the model is that of Zamolodchikov and Fateev (1985). We will not describe the algebra here but just mention that for $c=1$ one expects primary fields with the following values of Δ: 0, $\frac{1}{16}$, $\frac{1}{12}$, $\frac{1}{3}$, $\frac{9}{16}$, $\frac{3}{4}$, 1 and 3.

The degeracies along the conformal towers were not known till now. We are going to give them here:

$$\begin{aligned}
&[0]^{ZF}=\{0\} \qquad [1]^{ZF}=\{1\} \qquad [\tfrac{1}{16}]^{ZF}=[\tfrac{1}{16}]_1 \qquad [\tfrac{9}{16}]^{ZF}=[\tfrac{9}{16}]_1\\
&[\tfrac{1}{12}]^{ZF}=\bigoplus_{k\in\mathbf{Z}}(\tfrac{1}{12}(6k+1)^2) \qquad [\tfrac{1}{3}]^{ZF}=\bigoplus_{k\in\mathbf{Z}}(\tfrac{1}{3}(3k+1)^2)\\
&[\tfrac{3}{4}]^{ZF}=\bigoplus_{k\geqslant 0}(\tfrac{3}{4}(2k+1)^2) \qquad [3]^{ZF}=\bigoplus_{k\geqslant 1}(3k^2).
\end{aligned} \tag{30}$$

As noticed already earlier (von Gehlen and Rittenberg 1986b, Alcaráz 1986, Alcaráz and Lima Santos 1986) the Zamolodchikov-Fateev algebra occurs at $h=\frac{1}{3}$ and 3. The operator content for the two values of h is

$$\begin{aligned}
h=\tfrac{1}{3}: \qquad & D_{0,0}\oplus D_{1,0}=[0]^{ZF}\oplus[3]^{ZF}\\
& D_{0,1}\oplus D_{1,1}=[1]^{ZF}\oplus[3]^{ZF}\\
& D \qquad\quad\; =[\tfrac{3}{4}]^{ZF}\\
h=3: \qquad & D_{0,0}\oplus D_{1,0}=[0]^{ZF}\oplus[3]^{ZF}\oplus[\tfrac{1}{3}]^{ZF}\\
& D_{0,1}\oplus D_{1,1}=[1]^{ZF}\oplus[3]^{ZF}\oplus[\tfrac{1}{3}]^{ZF}\\
& D \qquad\quad\; =[\tfrac{3}{4}]^{ZF}\oplus[\tfrac{1}{12}]^{ZF}.
\end{aligned} \tag{31}$$

We conclude here the presentation of our results on the operator content of the Ashkin-Teller model with free boundary conditions. The more involved operator content for the other boundary conditions will be given in the following letter.

References

Alcaráz F C 1986 to be published

Alcaráz F C and Lima Santos A 1986 to be published

Ashkin J and Teller E 1943 *Phys. Rev.* **64** 178

Baake M, von Gehlen G and Rittenberg V 1987 *J. Phys. A: Math. Gen.* **20** 487

Berdshadski M A, Knizhnik V G and Teitelman M G 1985 *Phys. Lett.* **151B** 31

Boucher W, Friedman D and Kent A 1986 *Phys. Lett.* **172B** 316

Cardy J 1984 *Nucl. Phys.* B **240** 514

—— 1986 *Nucl. Phys.* B **275** 200

Corrigan E 1986 *Phys. Lett.* **169B** 259

Di Vecchia P, Petersen J L and Zheng HB 1985 *Phys. Lett.* **162B** 327

Eichenherr H 1985 *Phys. Lett.* **151B** 26

Friedan D and Shenker S 1986 to be published
Friedan D, Qiu Z and Shenker S H 1985 *Phys. Lett.* **151B** 37
Goddard P, Kent A and Olive D 1986 *Commun. Math. Phys.* **103** 105
Kac V 1979 *Lecture Notes in Physics* **94** 444
Kohmoto M, den Nijs M and Kadanoff L P 1981 *Phys. Rev.* B **24** 5229
Rittenberg V 1986 *Proc. Paris-Meudon Colloq., 22-26 September 1986* ed H J de Vega and N Sanchez (Singapore: World Scientific)
von Gehlen G and Rittenberg V 1986a *J. Phys. A: Math. Gen.* **19** L631
—— 1986b *J. Phys. A: Math. Gen.* **19** L1039
—— 1987 *J. Phys. A: Math. Gen.* **20** 227
Waterson G 1986 *Phys. Lett.* **171B** 77
Zamolodchikov A B and Fateev V A 1985 *Sov. Phys.-JETP* **62** 215